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BOUNDED VARIATIONS OF CONTINUOUS SOLUTIONS FOR A SYSTEM OF HYDRODYNAMIC EUQATIONS

/Following is the translation of an article by A. M. Molchanov entitled Ogranichennost' Variatsii Nepreryvnykh Resheniy Sistemy Uravneniy Gidrodinamiki (Bounded Variations of Continuous Solutions for a System of Hydrodynamic Equations) in Doklady Akademii Nauk SSSR (Reports of the Academy of Sciences USSR), 1959, Vol. 129, No. 6, pages 1257-1260./

The system of hydrodynamic equations has the aspect of

$$\frac{\partial u^\alpha}{\partial t} + \frac{\partial U^\alpha}{\partial x} = 0, \qquad (1)$$

where, using Lagrangian coordinates

$$u^{(1)} = u, \quad u^{(2)} = v, \quad u^{(3)} = E + \frac{u^2}{2}, \qquad (2)$$
$$U^{(1)} = p, \quad U^{(2)} = -u, \quad U^{(3)} = pu,$$

u is the velocity, v the specific volume, p the pressure, E = E (p, v) the internal energy density.

The purpose of this article is to prove the bounded character of variation in the solution of system (1). In connection herewith it is assumed that (a) the initial data have a limited variation, and (b) the solution is continuous in the range $0 \leqslant t \leqslant t_0, -\infty < x < +\infty$. The second assumption, strictly speaking, is superfluous, but it permits of performing the derivation by comparatively simple methods. Moreover, important features of the system of hydrodynamic equations, which appear distinctly, suggest the idea of segregating from the hyperbolical systems of equations of the type

$$\frac{\partial u^\alpha}{\partial t} + U^\alpha_\beta \frac{\partial u^\beta}{\partial x} = 0 \qquad (3)$$

(where U^α_β is a function of variables u^α) a special class of hydrodynamic-type equations.

These systems are distinguished by the requirement that they contain a function playing in certain respects the role of entropy in hydrodynamics. Since such a requirement pertains to the equality type, "entropy" systems form a rather small class within general hyperbolic systems. To

all these systems applies the proof of bonded variation solution as submitted below. "Entropy" systems occupy an intermediate position between linear systems, to which Cauchy's problem is applicable on the whole, and general systems (3) for which, apparently, the normal case is to convert the solutions into infinity even when t is finite for any smooth initial data equaling the constant outside the interval.

Let us indicate a way toward the proof. Derivations gain in clarity if carried out in a general way for system (3) with any number of equations. Our nearest aim is to write system (3) as an equivalent system of integral equations, from which one can evaluate $\int_{-\infty}^{+\infty}\left|\frac{\partial u^\alpha}{\partial x}\right|dx$. It can be achieved by doubling the number of unknown functions by introducing into the analysis the functions $v^\alpha = \partial u^\alpha/\partial x$ together with u^α. A system of equations for v^α is obtained by the differentiation of (3) with respect to x, and takes the form:

$$\frac{\partial v^\alpha}{\partial t} + \frac{\partial}{\partial x}(U^\alpha_\beta v_\beta) = 0. \qquad (4)$$

For the purpose of obtaining integral equations it is convenient to replace system (4) by a simpler diagonal system of other functions of φ^μ which are linear combinations of v^α with coefficients depending on u^α.

$$\varphi^\mu = l^\mu_\alpha v^\alpha. \qquad (5)$$

Functions of φ^μ are not total derivatives of any of the functions of u^α, as was the case with v^α. However, $\varphi^\mu dx = l^\mu_\alpha du^\alpha$ is a linear combination of differentials du^α, and it is convenient to write this expression as is done in thermodynamics, as $\delta\psi^\mu = l^\mu_\alpha du^\alpha$ and talk of an exact differential, keeping in mind, naturally, that it is not always possible to find $\psi^\mu(u^{(1)}, u^{(2)}, \ldots, u^{(n)})$ the differential of which would be equal to $l^\mu_\alpha du^\alpha$. Let us designate the inverse matrix to l^μ_α by m^α_ν so that $l^\mu_\alpha m^\alpha_\nu = \delta^\mu_\nu$. After simple operations we obtain

$$\frac{\partial \varphi^\mu}{\partial t} + \Lambda^\mu_\alpha \frac{\partial \varphi^\alpha}{\partial x} + \frac{\partial \Lambda^\mu_\alpha}{\partial x}\varphi^\alpha = C^\mu_{\alpha\beta}\varphi^\alpha\varphi^\beta, \qquad (6)$$

where

$$\Lambda^\mu_\nu = l^\mu_\gamma U^\gamma_\delta m^\delta_\nu, \qquad (7)$$

$$C^{\mu}_{\alpha\beta} = \left(\frac{\partial l^{\mu}_{\gamma}}{\partial u^{\delta}} - \frac{\partial l^{\mu}_{\delta}}{\partial u^{\gamma}}\right) \Lambda^{\gamma}_{\nu} m^{\delta}_{\alpha} m^{\nu}_{\beta}. \tag{8}$$

The existence of matrix l^{μ}_{γ}, leading U^{μ}_{γ} to the diagonal form Λ^{μ}_{ν} naturally follows from the hyperbolic nature of system (3). Matrix m^{δ}_{ν} simply consists of eigenvectors U^{μ}_{γ}, while vectors composing l^{μ}_{γ} form a system which is biorthogonal in relation to these eigenvectors. But since eigen vectors are determined with an accuracy only up to the multiplier, matrix l^{μ}_{γ} can be multiplied from the left by any diagonal matrix without changing matrix Λ^{μ}_{ν}.

This arbitrary assumption can be used for maximum simplification of the structure of tensor $C^{\mu}_{\alpha\beta}$. Let us try, for instance, to achieve equality to zero of the right-hand term in one of the equations of system (6). From (8) it appears that, generally speaking, (when $\Lambda^{\mu}_{\nu} \neq 0$) this will take place when and only when

$$\frac{\partial l^{\mu}_{\gamma}}{\partial u^{\delta}} - \frac{\partial l^{\mu}_{\delta}}{\partial u^{\gamma}} = 0, \tag{9}$$

i.e., only in the case when the exact differential $\delta\psi^{\mu} = l^{\mu}_{\alpha}\, du^{\alpha}$ is a total differential or, more exactly, when the exact differential has an integrating factor. For a system of two equations each exact differential has an integrating factor. In a system with a large number of equations, as a rule, none of the exact differentials has an integrating actor. It is all the more remarkable that hydrodynamic equations are an exception in this respect. The presence in this system of one integration factor for one of the exact differentials is equivalent to the known thermodynamic identity

$$dS = \frac{1}{T}(dE + p\, dv). \tag{10}$$

This observation justified the introduction of the term "entropy" for any exact differential being integrated.

Reverting to the general system (6), let us remark that the solution of such a system is already infinite at finite t_0, even when $t = 0$ and given any smooth initial conditions. This is clearly seen in the example of the following model system*

*This system is a model in the sense that it is unclear whether it is possible to indicate a type (3) - of even a type (1) - system, from which system (11) could be

$$\frac{\partial\varphi^{(1)}}{\partial t}+\frac{\partial\varphi^{(1)}}{\partial x}=\varphi^{(2)}\varphi^{(3)},\qquad \frac{\partial\varphi^{(2)}}{\partial t}=\varphi^{(3)}\varphi^{(1)},\qquad \frac{\partial\varphi^{(3)}}{\partial t}-\frac{\partial\varphi^{(3)}}{\partial x}=\varphi^{(1)}\varphi^{(2)} \tag{11}$$

with initial data $\varphi^{(1)}(x,0)=\varphi^{(2)}(x,0)=\varphi^{(3)}(x,0)=a(x)$ при $t=0$,
where a(x) is an even function of x, and is equal to unity on line segment (0, 2), smoothly decreasing to zero on line segment (2, 3), and equal to zero on half-line (3, ∞).
It is not difficult to verify that in the rectangle $-1\leqslant x\leqslant 1,\ 0\leqslant t<1$ the solution is given by Formula

$$\varphi^{(1)}(x,t)=\varphi^{(2)}(x,t)=\varphi^{(3)}(x,t)=\frac{1}{1-t}. \tag{12}$$

The phenomenon illustrated by the example of system (11) is substantially different from the phenomenon of "overturned front" ("gradient catastrophe") in one equation or in a system of two equations. The difference consists in the fact, that an increase in φ in one equation causes a narrowing of characteristics which compensates for increase in φ. This can be well observed in Eqs (6). If $C^{\mu}_{\alpha\beta}=0$, an increase in φ requires that $\partial\Lambda/\partial x<0$, which means that the characteristics are narrowing. It is not difficult to ascertain that the narrowing is just such as is required to assure that integral of absolute quantity φ is maintained. If, however, $C^{\mu}_{\alpha\beta}\neq 0$, it may happen that the terms $C^{\mu}_{\alpha\beta}\varphi^{\alpha}\varphi^{\beta}$ will cause an increase in φ that is not compensated by narrowing characteristics. Moreover, this may also happen in a segment of widened characteristics so that $\int_{-\infty}^{+\infty}|\varphi|\,dx$ will be growing due to both causes.

Thus, the segregation of terms $C^{\mu}_{\alpha\beta}\varphi^{\alpha}\varphi^{\beta}$, apparently, means the segregation of a noncompensated portion of the quadratic form -- i.e., of the portion which can convert the solution into infinity of finite t_0. Eqs (6) and Relations (8) distinctly show the difference in origin of terms of type $\frac{\partial\Lambda^{\mu}_{\alpha}}{\partial x}\varphi^{\alpha}=\frac{\partial\Lambda^{\mu}_{\alpha}}{\partial u^{\gamma}}lm^{\gamma}_{\beta}\varphi^{\alpha}\varphi^{\beta}$ and terms of type $C^{\mu}_{\alpha\beta}\varphi^{\alpha}\varphi^{\beta}$.

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obtained by the derivation described above. From formula (7) it follows that a type (3) system can be constructed with any constants Λ^{μ}_{ν}. In this case there still remains a free choice in the selection of matrix l^{μ}_{ν}, using which one can hope to obtain constants $C^{\mu}_{\alpha\beta}$. However, it is completely unclear whether a type (1) system can be constructed by this method.

The origin of the former is due to the variability of matrix U eigen values, and are automotically compensated by narrowing characteristics. The latter are caused by the rotation* of eigen vectors, and for them no compensating factors are awailable. In any case, these factors are absent if in a type (6) system the coefficients depending on $\underline{u}$ are changed for constants.

Turning to the proof of the basic theorem, let us first give an accurate definition of hydrodynamic-type equations.

Determination 1. The exact differential $\delta\psi^{\mu} = l^{\mu}_{a} du^{a}$ and the corresponding Equation in (6) are called "entropical if $\delta\psi^{\mu}$ is a total differential, i.e., when $C^{\mu}_{\alpha\beta}=0$ at any α and β values.

Determination 2. Equation system (3) is called a hydrodynamic-type system if it has at least one "entropy" while in non-entropy equations of system (6) the coefficients $C^{\mu}_{\alpha\beta}$ before the products of two non-entropical terms $\varphi^{\alpha}\varphi^{\beta}$ are equal to zero.

Remark. To be a system of hydrodynamic type, it suffices if a system of three equation has "entropy". Indeed, it can be achieved that tensor $C^{\mu}_{\alpha\beta}$ has the necessary structure by proper selection of the l^{μ}_{a} matrix. To this end, it is sufficient if the two coefficients C^{1}_{12} and C^{2}_{12} are converted to zero. Equalities $C^{1}_{12}=0$ and $C^{2}_{12}=0$ give two length conditions of own nonentropy vectors.

Let a hydrodynamic-type system now be given. We write for it system (6) renumbering, if required, the equation in such a manner that the entropy equations come first

*Something similar probably occurs even in system of ordinary equations $dr/dt = A(t)r$. Assuming $A = U\Lambda U^{-1}$, where Λ is a constant triangular matrix with negative eigen values, while a $U = e^{St}$, where S is a constant antisymmetric matrix. Although all eigen values of matrix A are negative, one can - according to Lyapunov - judge about the stability of the system only if U is rotated slowly, i.e., at low S. If, however, one takes

$$\Lambda = \begin{pmatrix} -1 & 7 \\ 0 & -2 \end{pmatrix}, \text{ a } S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the system becomes unstable.

$$\frac{\partial \varphi^\mu}{\partial t} + \frac{\partial}{\partial x}(\Lambda^\mu_\alpha \varphi^\alpha) = 0, \quad 1 \leqslant \mu \leqslant n_0; \tag{13}$$

$$\frac{\partial \varphi^\nu}{\partial t} + \frac{\partial}{\partial x}(\Lambda^\nu_\alpha \varphi^\alpha) = p^\nu_\gamma \varphi^\gamma + q^\nu, \quad n_0 + 1 \leqslant \nu \leqslant n. \tag{14}$$

Here the coefficients p^ν_γ are expressed through "entropy" terms φ^μ linearly and q^ν quadratically.

Thus, for hydrodynamic-type systems "splitting off" of entropy equations takes place, which (if one considers functions $C^\alpha_{\alpha\beta}$ and Λ^μ_ν as given, an admissible assumption for a priori evaluations) are integrated independently and give solutions for which $\int\limits_{-\infty}^{+\infty} |\varphi^\mu(x, t)|\,dx$ is univormly bounded at any $t > 0$.

For nonentropy terms, a system of linear equations with coefficients depending on entropy terms is formed. For such system integrals $\int\limits_{-\infty}^{+\infty} |\varphi^\nu(x, t)|\,dx$ are, in general, no more bounded. However, they increase at a rate not faster than $\exp\left(\int\limits_0^t a\,dt\right)$, where $a(t) = \max\limits_{1 \leqslant \mu \leqslant n_0} \int\limits_{-\infty}^{+\infty} |\varphi^\mu(x, t)|\,dx$.

This evaluation can be proved, for example, by writing Eqs (14) as integral equations along the corresponding characteristics and applying to the integral equations obtained the method of consecutive approximations.

Remark. In this proof substantial use is made of the fact that for any system $C^\mu_{\alpha\alpha} = 0$ at any μ and α. This affirmation derives directly from (8) if one takes into consideration that Λ^γ_i is a diagonal matrix. It leads to a simple evaluation of q^ν terms

$$\int\limits_0^{t_1}\int\limits_{-\infty}^{+\infty} |q^\nu(x, t)|\,dx\,dt \leqslant \text{const} \max_{\substack{1 \leqslant \mu \leqslant n_0 \\ 0 \leqslant t \leqslant t_1}} \left[\int\limits_{-\infty}^{+\infty} |\varphi^\mu(x, t)|\,dt\right]^2.$$

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